A Local Discontinuous Galerkin Method for Eddy Current Field Analysis in High-speed Moving Conductors

Yanpu Zhao, Xiu Zhang, S. L. Ho, and W. N. Fu
Department of Electrical Engineering, The Hong Kong Polytechnic University, Kowloon, Hong Kong
eeypzhao@polyu.edu.hk, 09901605R@polyu.edu.hk, eeslho@polyu.edu.hk, eewnfu@polyu.edu.hk

Abstract — A numerical method for analysing eddy current magnetic field in high-speed moving conductors using local discontinuous Galerkin (LDG) method is presented. A typical numerical example is chosen to illustrate the accuracy and non-oscillatory nature of the method for a convection dominated case. The simulation results using the proposed algorithm are validated using standard finite element method (FEM).

I. INTRODUCTION

When electromagnetic field computation involves high-speed moving conductors, eddy currents due to the movement of conductors should be taken into account and the common mathematical simulation model is usually derived from convection-dominated equation [1]. The standard finite element method (FEM) cannot solve high speed moving conductor problems effectively unless very fine meshes, which are computationally expensive to process, are introduced [2-3]. Both the mixed FEM [4] and operator splitting method [5] introduce some upwind mechanism to treat the first-order term in the equation. Reference [6] adopts the finite analytic element method and recently [7] uses the multiscale combined radial basis function collocation method to derive the numerical solution of the problem.

During this decade, the discontinuous Galerkin (DG) method for the numerical solution of hyperbolic partial differential equations (PDEs) and related local discontinuous Galerkin (LDG) method for parabolic PDEs and elliptic PDEs have been recognized to be effective numerical tools to address convection-dominated problems [8-10]. Moreover, these methods can be realized in high-order elements easily. They can also treat the convection term properly by suitably defining the numerical flux in the scheme. In this paper, the formulation of a LDG method is presented for the steady state convection dominated problem and numerical tests are carried out to illustrate the advantages of this method.

II. LDG SCHEME

Consider the steady state convection-diffusion boundary value problem in Cartesian coordinates:

$$\begin{aligned} &\Omega: \nabla^2 u + \nabla \cdot \vec{f}(u) + k^2 u = 0 \\ &\partial\Omega: u = g \end{aligned}, \tag{1}$$

where Ω is bounded in R^d; d=1, 2, 3, $\vec{f}, g$ are known functions and u is the unknown function to be solved numerically. In this paper, only the linear case is discussed, that is, $\vec{f}$ is linear about u. However, the magnitude of the Jacobian $(\partial / \partial u)\vec{f} \cdot \vec{n}$ can be large. It is well known that for such case the solution will have a steep boundary layer near $\partial\Omega$, which is difficult to simulate accurately by standard FEM [2-3].

For simplicity of discussion, define the k-th order discontinuous finite element space as:

$$V_h = \{u \in L^2(\Omega) : u|_K \in P^k(K), \forall K \in T\}. \tag{2}$$

where Ω is the bounded region in R^d; T is the triangulation of Ω; h is the maximum side length of the triangulation.

Take any two neighboring elements K^+ and K^- from T, and the common side $\gamma = \partial K^+ \bigcap \partial K^-$; $\vec{n}^+$ and $\vec{n}^-$ are, respectively, the unit outward normal vectors from the interior of K^+ and K^- at any point of γ, as shown in Fig. 1. Let $w^{\pm}$ be the respective trace of w from the interior of $K^{\pm}$. Define $\{\{\cdot\}\}$, $[[\cdot]]$ the average and jump of related function at $x \in \gamma$:

$$\begin{aligned} &\{\{u\}\} = (u^+ + u^-)/2, \quad \{\{\vec{q}\}\} = (\vec{q}^+ + \vec{q}^-)/2 \\ &[[u]] = u^+\vec{n}^+ + u^-\vec{n}^-, \quad [[q]] = \vec{q}^+\vec{n}^+ + \vec{q}^-\vec{n}^- \end{aligned}. \tag{3}$$

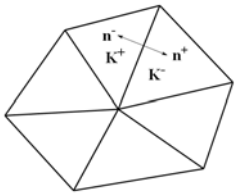

Fig. 1. Illustration of the notations.

Rewrite the partial differential equation in (1) into

$$\begin{cases} \vec{q} = \nabla u \\ \nabla \cdot \vec{q} + \nabla \cdot \vec{f}(u) + k^2 u = 0 \end{cases}. \tag{4}$$

Then the LDG scheme for (1) reads [11-12]

$$\begin{cases} \int_K \vec{q}_h \cdot \vec{w} dV + \int_K u_h \nabla \cdot \vec{w} dV = \int_{\partial K} \hat{u}_h \vec{w} \cdot \vec{n} ds \\ \int_K \vec{q}_h \cdot \nabla v dV + \int_K \vec{f}(u_h) \cdot \nabla v dV - \int_K k^2 uv dV, \\ = \int_{\partial K} v\hat{q}_h \cdot \vec{n} ds + \int_{\partial K} v\hat{f} \cdot \vec{n} ds \end{cases} \tag{5}$$

for any $(\vec{w}, v) \in V_h^2 \times V_h$; where $\hat{u}_h$, $\hat{q}_h$ and $\hat{f}$ are the numerical fluxes which are defined by

$$\begin{aligned} \hat{f} \cdot \vec{n} &= \frac{1}{2}\left(\vec{f}(u^-) \cdot \vec{n} + \vec{f}(u^+) \cdot \vec{n} - \beta[[u]]\right) \\ \hat{q}_h &= \{\{\vec{q}_h\}\} - C_{11}[[u_h]] - \vec{C}_{12}[[\vec{q}_h]] \\ \hat{u}_h &= \{\{u_h\}\} + \vec{C}_{12}[[u_h]] \end{aligned}, \tag{6}$$

in the interior of the domain by

$$\begin{aligned} \hat{q}_h &= \vec{q}_h{}^{+} - C_{11}(u_h^{+} - g)\vec{n}, \text{on } \partial\Omega \\ \hat{u}_h &= g, \text{on } \partial\Omega \end{aligned} \quad , \tag{7}$$

on the boundary of the domain Ω , where β is an estimation of the biggest eigen value of the Jacobian $(\partial / \partial u)\vec{f} \cdot \vec{n}$, $C_{11} = O(1/h)$ and $\vec{C}_{12}$ is a vector in R^2 of length 1/2. The definition of the numerical flux $\hat{f} \cdot \vec{n}$ on $\partial\Omega$ can be found in [12]. Note that the numerical flux $\hat{f}$ is nothing but the local Lax-Friedrichs numerical flux and it can be taken in other forms [13].

III. NUMERICAL RESULTS

A classical one-dimensional convection-diffusion equation defined in [0,1] is set as

$$\begin{cases} u'' + pu' + k^2 u = 0 \\ u|_{x=0} = 0, u|_{x=1} = 1 \end{cases}, \tag{8}$$

to verify the local discontinuous Galerkin method proposed in this paper.

Take k=0, then the exact solution for (8) is

$$u = \frac{e^{-px} - 1}{e^{-p} - 1}. \tag{9}$$

In the following, (8) and (9) are solved numerically using the LDG method. For small p, the convergence order of the L^2 error by the LDG method using k-th order polynomial space is given for k=1, 2, 3, 4 and 5. For large p, the numerical result computed by the LDG method gives a good illustration of the advantage of this method.

Table I gives the L^2 error of the numerical solution u_h to the exact solution u in (9) for p=-10. It can be seen clearly that when the k-th order polynomial space is used as the test and the trial space for the LDG method, the L^2 error order of convergence is k+1, which is optimal according to the classical finite element analysis result [3].

TABLE I
CONVERGENCE ORDER FOR P=-10 OF THE LDG METHOD

k	# of cells	L^2 error of u - u_h	L^2 error order
1	20	2.7983E-003	-
	40	7.2823E-004	1.9421
	80	1.8535E-004	1.9742
	160	4.6727E-005	1.9879
2	20	1.1869E-004	-
	40	1.5323E-005	2.9533
	80	1.9411E-006	2.9808
	160	2.4409E-007	2.9914
3	20	3.7498E-006	-
	40	2.4104E-007	3.9595
	80	1.5231E-008	3.9842
	160	9.5643E-010	3.9932
4	20	9.4571E-008	-
	40	3.0319E-009	4.9631
	80	9.5657E-011	4.9862
	160	3.0013E-012	4.9942
5	20	1.9825E-009	-
	40	3.1732E-011	5.9653
	80	5.0007E-013	5.9877
	160	9.6945E-015	5.6888

For large p, the boundary value problem (8) is a convection-dominated problem in which the first order term needs to be treated properly when designing numerical schemes. It can be seen that the LDG method gives a good numerical solution even when p is large.

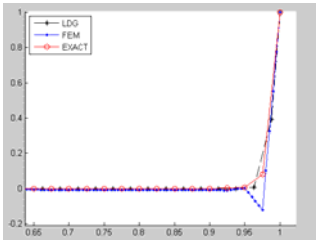

Fig. 2. p=-100, number of cells=40, numerical solution by the LDG method with k=1 and the linear FEM versus the exact solution.

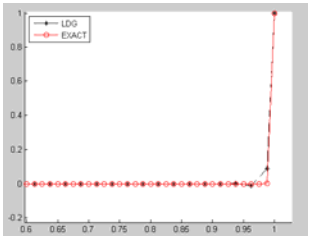

Fig. 3. p=-500, number of cells=40, numerical solution by the LDG method with k=1 versus the exact solution.

IV. REFERENCES

[1] N. Burais, N. Foggia and A. Nicolas, "Numerical solution of eddy currents problems including moving conducting parts," *IEEE Trans. Magn.*, vol. 20, no. 5, pp. 1995–1997, 1984.

[2] T. Ikeda, *Maximum principle in finite element models for convection-diffusion phenomena*, North-Holland publishing company, pp. 1-8, 1983.

[3] J. E. Flaherty, *Finite element analysis*, Lecture notes: Spring 2000, 2000.

[4] C. N. Dawson, "Analysis of an upwind-mixed finite element method for nonlinear contaminant transport equations," *SIAM J. Numer. Anal.*, vol. 35, no. 5, pp. 1709-1724, 1998.

[5] M. Remesikova, Numerical solution of two-dimensional convection–diffusion–adsorption problems using an operator splitting scheme, *Applied Mathematics and Computation*, vol.184, pp. 116–130, 2007.

[6] D. Chen, K. R. Shao, H. Yu and J. D. Lavers, "A novel finite analytic element method for solving eddy current problems with moving conductors," *IEEE Trans. Magn.*, vol. 37, no. 5, pp. 3150-3154, 2001.

[7] Y. Zhang, G. Y. Yang, K. R. Shao, Y. G. Guo, J. G. Zhu and J. D. Lavers, "Multiscale Combined Radial Basis Function Collocation Method for Eddy Currents Analysis in High-Speed Moving Conductors," *IEEE Trans. Magn.*, vol. 45, no. 10, pp. 3973-3976, 2009.

[8] B. Cockburn and C. W. Shu, "TVB Runge-Kutta local projection discontinuous Galerkin nite element method for conservation laws V: multidimensional systems," *J. Comput. Phys.*, vol. 141, no.2, pp. 199-224, 1998.

[9] B. Cockburn and C. W. Shu, "The local discontinuous Galerkin method for time-dependent convection-diffusion systems," *SIAM J. Numer. Anal.*, vol.35, no. 6, pp. 2440-2463, 1998.

[10] P. Castillo, B. Cockburn, I. Perugia and D. Schotzau, "An A priori error analysis of the local discontinuous Galerkin method for elliptic problems," *SIAM J. Numer. Anal.*, vol.38, no. 5, pp. 1676-1706, 2000.

[11] Y. P. Zhao, "Study on the local discontinuous Galerkin method for the Poisson equation in curvilinear coordinates", *Thesis*, Nankai University, 2009.

[12] C. W. Shu, "Discontinuous Galerkin Method," unpublished.

[13] J. X. Qiu, B. C. Khoo and C. W. Shu, "A numerical study for the performance of the Runge-Kutta discontinuous Galerkin method based on different numerical fluxes," Journal of Comp. Phys., vol. 212, pp. 540-565, 2006.